Math 122 / Midterm 2

Wednesday, November 16
1 hour, closed book, no aids

1. True or false? (No justification necessary.)

(a) If V is a finite dimensional real vector space with a dot product, then any linearly independent set of vectors is orthogonal.

(b) Any commutator $[g,h]$ in the group $GL_2(\mathbb{R})$ lies in the subgroup $SL_2(\mathbb{R})$.

(c) Any element in the non-trivial coset of $SO(2)$ in $O(2)$ has order 2.

(d) If G acts on a set S of order n then G is isomorphic to a subgroup of S_n.

(e) If H is a subgroup of the finite group G, then the index of the normalizer $N(H)$ in G is the number of distinct conjugates of the subgroup H.

(f) If V is a vector space of dimension n over the field F, then any set of $n+1$ distinct vectors forms a spanning set for V.

(g) The element $g = (1234679)(85)$ in S_9 satisfies $g^7 = (58)$.

(h) Any group of order 21 is non-abelian.

2. Let G be a finite group of order mp^n where p is prime and m is prime to p.

(a) State Sylow's theorems on subgroups H of G of order p^n.

(b) Prove or disprove: any finite group of order 35 is cyclic.

3. Let G be a finite p-group which acts on a finite set S.

(a) Define the set T of fixed points of G.

(b) Show that $\#S \equiv \#T \pmod{p}$.

4. **(a)** Define the orthogonal group $O(n)$ as a subgroup of matrices A in $GL_n(\mathbb{R})$.

(b) Define an orthogonal linear transformation T of a real inner product space V of dimension n.

(c) Show that the matrix of T, with respect to an orthonormal basis, is in the subgroup $O(n)$ defined in part a.

(d) Show that any eigenvalue of an orthogonal transformation is equal to ± 1.

5. **(a)** Define the dihedral group D_n of order $2n$. For what values of $n \geq 1$ is D_n an abelian group?

(b) Show that the Sylow 2-subgroup of the symmetric group S_4 is isomorphic to the dihedral group D_4.